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General Quasiparticle Propagator and Mass Dependence of Condensates in Degenerate Colorsuperconductivity

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Abstract:

The most general quasiparticle propagator in dense quark matter is derived for equal mass quarks. Specialized to an NJL model this propagator includes one new condensate, Δ_3, in addition to the usual CFL condensate, Δ_1. The gap equation is solved in two NJL models and the dependence of the condensates on the mass of the quarks is presented. The results are shown to differ from those obtained by neglecting Δ_3, especially for smaller values of Δ_1. The methods used in this paper can be generalized to the physical case where only the strange quark is significantly massive.

1 Introduction

The physics of strongly interacting matter at high densities and low temperatures has been the subject of much research in recent years. It has long been known[1, 2] that at sufficiently high densities a system of quarks should form a condensate of Cooper pairs which breaks the $SU_C(3)$ symmetry and becomes a color superconductor. The formation of the condensates leads to gaps in the quasi-particle mass spectrum[1]. The authors of [1, 2] estimated that the gaps were of the order of $\Delta \sim 10^{-3}\mu$ where μ is the quark chemical potential. Recently it was shown at realistic values of μ in an instanton induced NJL model that gaps of the order of μ could be obtained[3]. This stimulated a great deal of research in the ensuing years and has resulted in a proliferation of predicted superconducting ground states. These states may be realized in the cores of neutron stars and lead to observable effects[4].

It is widely accepted that the color superconducting ground state at asymptotic densities is the Color Flavor Locked (CFL) state[5]:

$$\langle q^i_\alpha C\gamma^5 q^j_\beta\rangle = \Delta_{\bar{3}\bar{3}}(\delta^i_\alpha\delta^j_\beta - \delta^i_\beta\,\delta^j_\alpha) + \Delta_{66}(\delta^i_\alpha\delta^j_\beta + \delta^i_\beta\,\delta^j_\alpha), \tag{1}$$

where the Greek indices are color indices, the Latin indices are flavor indices and the $\bar{3}$ and 6 subscripts refer to anti-triplet or sextet configurations in color and flavor spaces respectively. At lower densities ($\mu \sim m_s$), it is likely that the ground state is a superconducting state involving the condensation of Cooper pairs in the $u-d$ sector only (2SC). Between these two limits, two new phases have recently been predicted: Crystalline Color Superconductivity[6] and CFL with meson condensation[7]. These predictions, while not necessarily at odds with one another, indicate that the transition region between the CFL state and the 2SC state is not completely understood.

Including the strange quark mass in the Nambu-Gorkov gap equation introduces two sets of complications: 1) massive quarks means that there are 4 new types of Dirac structures allowed for the condensates, and coupling between the condensates; 2) the fact that the strange quark is different from the other quarks means that condensates involving the strange quark should be different from those with zero strangeness.

In order to understand the implications of these two complications it is useful to separate them and understand them individually before tackling

[1]The terms gap and condensate are used interchangeably.

the full problem. In a previous paper by this author[8] the second problem of non-degenerate quarks was studied in the case where the quarks are all massless but the strange quark is given a different chemical potential than the other two quarks. In this paper we concentrate on the first complication by considering the problem of equal mass quarks.

The results of this paper are exact for the $N_f = 2$ and for the the 2SC phase with $N_f = 3$. The results are an approximation to the $N_f = 3$ CFL case. We do not explicitly deal with the color flavor structure because it cancels out.

The most general quasiparticle propagator is presented for the case of equal mass quarks. This propagator is then specialized to the NJL model where it was found that there is only one new condensate, Δ_3 in addition to the CFL condensate, Δ_1.

We solve the gap equation in the simplest NJL model and present results for the most general set of condensates as a function of the quark mass. The results are compared with results obtained neglecting the effect of the new condensate Δ_3. It is shown that the inclusion of Δ_3 alters the results especially for lower values of Δ_1. These results are relevant to the analysis of [22] where the gap equation is solved ignoring Δ_3.

We also do the same analysis in the NJL model which has the color flavor structure of single gluon exchange. The results in this model are also altered by the inclusion of Δ_3 especially for lower values of Δ_1.

These results show the importance of using the most general form of the quasiparticle propagator in the solution of the gap equations for massive but degenerate quarks. The results are most significant for smaller values of the the CFL condensate.

The NJL analysis is a low energy motivated model. The analysis can be repeated using perturbation theory in order to compare with a model valid at higher densities and energies.

The methods used in this paper are a significant new result on their own. This is because they can be generalized to determine the general quasiparticle propagator and the general set of gap equations for the physical case where the up and down quarks are essentially massless but the strange quark is massive.

2 Basis for the Condensates

The CFL condensate has the Dirac structure γ_5 and is the only spin zero condensate in the massless case. The most general set of spin zero condensates is given in Appendix B and includes eight different possible structures. In an NJL model the structures involving $\gamma \cdot \hat{k}$ automatically vanish leaving four possible Dirac structures. Later in the paper we will see that, in the NJL analysis, there is only one new condensate, Δ_3 with the Dirac structure: $\gamma_5\gamma_0$.

The analysis could be carried out in a basis of Dirac structures, but the equations would be very complicated. A crucial step in finding the general quasiparticle propagator and gap equations is choosing a basis of Dirac structures which simplifies the analysis enough to make it tractable.

In the massless case an arbitrary condensate matrix can be decomposed using a basis of projection operators[15]:

$$P_h^e(\vec{k}) = \Lambda^e(\vec{k}) P_h(\hat{k}), \qquad e, h = \pm 1 \tag{2}$$

which are products of positive and negative energy projectors:

$$\Lambda^e(\vec{k}) = \frac{1 + e\,\gamma_0 \vec{\gamma} \cdot \vec{k}}{2} \qquad e = \pm 1 \tag{3}$$

and positive and negative helicity projectors:

$$P_h(\hat{k}) = \frac{1 + e\,\gamma_5 \gamma_0 \vec{\gamma} \cdot \vec{k}}{2} \qquad h = \pm 1 \tag{4}$$

as:

$$\Delta = \sum_{e,h,\pm 1} \Delta_h^e \, P_h^e \tag{5}$$

Similarly all objects, such as the bare quark and the quasiparticle propagator can be written in this basis. The basis of projection matrices is extremely useful since it reduces products of these objects to the simplest possible form.

In the case of massive quarks, however, things are more complicated. The energy projector can be generalized to[15]:

$$\Lambda^e(\vec{k}) = \frac{1 + e\,\left(\beta \gamma_0 \vec{\gamma} \cdot \vec{k} + \alpha \gamma_0\right)}{2} \qquad e = \pm 1 \tag{6}$$

where $\beta \equiv |\vec{k}|/E_k$ and $\alpha \equiv m/E_k$.

A set of analagous operators in the massive case given in [15] is:

$$P^e_{c\,h}(\vec{k}) = P_c\,\Lambda^e(\vec{k})P_h(\hat{k}), \qquad e, h = \pm 1 \quad c = r, l \tag{7}$$

where the new operator in the product is the chirality projector

$$P_c = \left(\frac{1 + c\,\gamma_5}{2}\right) \qquad c = r, l \tag{8}$$

which projects onto right and left handed spinors. In what follows I will use the definition $r = +$ and $l = -$ in analogy with the other projectors which will simplify a lot of the formulas below.

These operators are really quasiprojectors with the general product rule:

$$P^e_{ch}(k)P^{e'}_{c'h'}(k) = \delta_{ee'}\delta_{cc'}\delta_{hh'}P^e_{ch}(k) - \frac{1}{2}e\,c\,e'c'(1 - e'c\,h\,\beta)\,\delta_{hh'}\,P^{e'}_{ch}(k) \tag{9}$$

The objects are not projectors because chirality and energy projectors do not commute:

$$[P_c, \Lambda^e] = -e\,c\,\alpha\;\gamma_0\gamma_5 \tag{10}$$

This basis is complete and the general gap matrix can be written using this basis[15]:

$$\Delta = \sum_{e,c,h,\pm 1} \Delta^e_{c\,h}\;P^e_{c\,h} \tag{11}$$

but this basis still involves a lot of complications.

We can define a new basis of true projectors and nilpotent operators:

$$P^e_h(k) = \sum_{c=-1,1} P^e_{ch}(k) = P_h(k)\;\Lambda^e(k) \tag{12}$$

$$Q^e_h(k) = \sum_{c=-1,1} (e\,c\,h - \beta)P^e_{ch}(k) = (e\,h\,\gamma^5 - \beta)P^e_h(k) \tag{13}$$

that satisfy the following relations:

$$\begin{aligned}
P^e_h(k)P^{e'}_{h'}(k) &= \delta_{ee'}\delta_{hh'}P^e_h(k) && (14)\\
Q^e_h(k)P^{e'}_{h'}(k) &= \delta_{ee'}\delta_{hh'}Q^e_h(k) && (15)\\
P^e_h(k)Q^{e'}_{h'}(k) &= \delta_{-ee'}\delta_{hh'}Q^{-e}_h(k) && (16)\\
Q^e_h(k)Q^{e'}_{h'}(k) &= -\alpha^2\delta_{-ee'}\delta_{hh'}P^{-e}_h && (17)
\end{aligned}$$

$P_h^e(k)$ is an hermitian operator and $(Q_h^e(k))^\dagger = -Q_h^{-e}(k)$.

In the chiral limit ($\alpha \to 0$,$\beta \to 1$) $P_h^e(k)$ are identical to the massless projectors and $Q_h^e(k)$ vanish.

Notice that the operators with different helicities are completely decoupled.

The multiplication table for each helicity looks like:

$$
\begin{array}{cc}
 & \begin{array}{cccc} P_h^+ & Q_h^+ & P_h^- & Q_h^- \end{array} \\
\begin{array}{c} P_h^+ \\ Q_h^+ \\ P_h^- \\ Q_h^- \end{array} &
\begin{pmatrix}
P_h^+ & 0 & 0 & Q_h^- \\
Q_h^+ & 0 & 0 & -\alpha^2 P_h^- \\
0 & Q_+^+ & P_h^- & 0 \\
0 & -\alpha^2 P_h^+ & Q_h^- & 0
\end{pmatrix}
\end{array}
\tag{18}
$$

This set of eight operators is the set of operators on the space of solutions of the Dirac equation which gives the most sparse multiplication table. One would prefer of course, a basis of projection operators as in the massless case. However, such a set does not exist. Dirac spinors are 4 component complex vectors and as such live in an $\mathcal{R}^8$. One can definitely find eight orthogonal projectors in this space. However, Dirac spinors must be solutions of the Dirac equation which places constraints on the components. Therefore you cannot have 8 orthogonal projectors onto the space of solutions of the Dirac equation. On the other hand you need 8 operators to form a complete set with which to contruct all 8 independent Dirac structures.

In the massless case, the left and right handed spinors decoupled so that each was a two component complex spinor which therefore lives in $\mathcal{R}^4$. It is therefore possible to find a set of 4 orthogonal projectors on this space.

The operators given above have the following represention as 2×2 matrices.

$$
P_h^+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad P_h^- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{19}
$$

$$
Q_h^+ = \begin{pmatrix} 0 & 0 \\ -\alpha & 0 \end{pmatrix} \quad Q_h^- = \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix} \tag{20}
$$

This supports the reasoning above that there are only 4 independent projectors since the operators for each handedness operate on a two dimensional space. This representation will be useful in what follows.

This set of operators is complete as is shown by the following relations:

$$I = \sum_{e,h=-1,1} P_h^e(k) \tag{21}$$

$$\gamma^5\gamma^0\gamma\cdot\hat{k} = \sum_{e,h=-1,1} h\, P_h^e(k) \tag{22}$$

$$\alpha\,\gamma\cdot\hat{k} = -\sum_{e,h=-1,1} Q_h^e(k) \tag{23}$$

$$\alpha\,\gamma^5\gamma^0 = \sum_{e,h=-1,1} h\, Q_h^e(k) \tag{24}$$

$$\gamma^0\gamma\cdot\hat{k} = \sum_{e,h=-1,1} e\,(Q_h^e(k) + \beta\, P_h^e(k)) \tag{25}$$

$$\gamma^5 = \sum_{e,h=-1,1} e\,h\,(Q_h^e(k) + \beta\, P_h^e(k)) \tag{26}$$

$$\alpha\,\gamma^0 = \sum_{e,h=-1,1} e\,(\alpha^2 P_h^e(k) - \beta\, Q_h^e(k)) \tag{27}$$

$$\alpha\,\gamma^5\gamma\cdot\hat{k} = -\sum_{e,h=-1,1} e\,h\,(\alpha^2 P_h^e(k) - \beta\, Q_h^e(k)) \tag{28}$$

It should be noted that these equations are still valid in the chiral limit (albeit half of them are trivial).

The gap matrix is decomposed in terms of these operators as:

$$\Delta = \sum_{e,h=-1,1} \left(\beta\,\xi_h^e + \alpha^2\,\psi_h^e\right) P_h^e(k) + (\xi_h^e - \beta\,\psi_h^e)\, Q_h^e(k) \tag{29}$$

$$\Delta^\dagger = \sum_{e,h=-1,1} \left(\beta\,\xi_h^e + \alpha^2\,\psi_h^e\right) P_h^e(k) - (\xi_h^e - \beta\,\psi_h^e)\, Q_h^{-e}(k) \tag{30}$$

It should be noted that due to the vanishing of $Q_h^e(k)$ in the chiral limit, this decomposition of Δ gives no restriction (other than finiteness) on the coefficient, $(\xi_h^e - \beta\,\psi_h^e)$, in the chiral limit.

All objects that we are concerned with in this analysis can therefore be constructed using this basis. Their sparse multiplication table simplifies much of the analysis in the rest of the paper. In the next section we derive the general quasiparticle propagator using this basis.

3 General Quasiparticle Propagator

The quasiparticle propagator is determined by the equation[15]:

$$G^{\pm} \equiv \left\{ [G_0^{\pm}]^{-1} - \Delta^{\mp} G_0^{\mp} \Delta^{\pm} \right\}^{-1}. \tag{31}$$

where $\Delta^+ \equiv \Delta$, $\Delta^- \equiv \gamma_0 \Delta^+ \gamma_0$ and G_0^- is the bare antiparticle propagator:

$$\begin{aligned}
G_0^-(k) &= \left(\gamma^\nu k_\nu - \mu\gamma^0 - M\right)^{-1} = \frac{\gamma^\nu k_\nu - \mu\gamma^0 + M}{(k_0-\mu)^2 - E_k^2} \\
&= \gamma^0 \frac{\left((k_0-\mu) - E_k \beta \gamma^0 \gamma \cdot \hat{k} + E_k \alpha \gamma^0\right)}{(k_0-\mu)^2 - E_k^2} \\
&= \gamma^0 \frac{(k_0-\mu) + E_k \sum_{e,h=-1,1} e\ [(\alpha^2-\beta^2) P_h^e(k) - 2\beta\, Q_h^e(k)]}{(k_0-\mu)^2 - E_k^2}
\end{aligned} \tag{32}$$

If we define:

$$\begin{aligned}
\Omega &= \left[G^+(k) \left((k_0-\mu)\gamma^0 - \gamma \cdot \vec{k} + M\right)^{-1} \right]^{-1} \\
&= \sum_{e,h=-1,1} (A_h^e P_h^e(k) + B_h^e Q_h^e(k)) = \sum_{h=-1,1} \Omega_h
\end{aligned} \tag{33}$$

the coefficients are:

$$\begin{aligned}
A_h^e &= (k_0)^2 - \epsilon^e(\xi_h^e, \psi_h^e)^2 - \frac{2\, e\, \alpha_k^2\, E_k}{(k_0-\mu) - eE_k} (\psi_h^e)^2 \\
B_h^e &= \xi_h^{-e} \psi_h^e - \xi_h^e \psi_h^{-e} + \frac{2\, e\, E_k}{(k_0-\mu) + eE_k} \xi_h^e\, \psi_h^{-e}
\end{aligned} \tag{34}$$

using the notation:

$$\epsilon^{\pm}(\xi, \psi) = \left[(E_k \mp \mu)^2 + \xi^2 + \alpha_k^2\, \psi^2 \right]^{1/2} \tag{35}$$

Using the 2×2 matrix representation of P_h^e and Q_h^e given above we find:

$$\Omega^{-1} = \sum_{e,h=-1,1} \frac{1}{\det \Omega_h} \left(A_h^{-e} P_h^e(k) - B_h^e Q_h^e(k) \right) \tag{36}$$

where:

$$
\begin{aligned}
\det \Omega_h &= k_0^4 - 2k_0^2 \left(E_k^2 + \mu^2 + (\xi_h^+)^2 + (\xi_h^-)^2 + \alpha^2 (\psi_h^+)^2 + \alpha^2 (\psi_h^-)^2 \right) \\
&+ 2E_k k_0 \left((\psi_h^+)^2 - (\psi_h^-)^2 \right) \\
&+ \Big[(E_k - \mu)^2 (E_k + \mu)^2 + (E_k + \mu)^2 (\xi_h^+)^2 + (E_k - \mu)^2 (\xi_h^-)^2 \\
&- \alpha^2 (E_k - \mu)(E_k + \mu) \left((\psi_h^+)^2 - (\psi_h^-)^2 \right) + \left(\xi_h^+ \xi_h^- + \alpha^2 \psi_h^+ \psi_h^- \right)^2 \Big]
\end{aligned} \tag{37}
$$

and we have used the definition:

$$
E_k = \sqrt{|\vec{k}|^2 + m^2} \tag{38}
$$

The roots of this quartic equation in k_0 can be found in an appendix(not yet).

Now we have that:

$$
G^+(k) = \Omega^{-1} \left((k_0 - \mu)\gamma^0 - \gamma \cdot \vec{k} + M \right) \tag{39}
$$

The completely general propagator is:

$$
\begin{aligned}
G^+(k) &= \sum_{e,h=-1,1} \frac{\alpha(e\ k_0 - e\ \mu + E_k))(k_0^2 - (\omega_1)_h^e) - \alpha(e\ k_0 - e\ \mu - E_k)\,(\beta\ \upsilon_h^e)\,\psi_h^{-e}}{\det \Omega_h} P_h^e \\
&- \sum_{e,h=-1,1} \frac{\beta/\alpha(e\ k_0 - e\ \mu - E_k)(k_0^2 - (\omega_2)_h^e) + \alpha(e\ k_0 - e\ \mu - E_k)\eta_h^e \psi_h^{-e}}{\det \Omega_h} Q_h^e
\end{aligned}
$$

where:

$$
\begin{aligned}
(\omega_1)_h^e &= (E_k + e\ \mu)^2 + \xi_h^{-e} \chi_h^{-e} && (40) \\
(\omega_2)_h^e &= (E_k + e\ \mu)^2 + \frac{1}{\beta}\ \xi_h^{-e} \phi_h^{-e} && (41)
\end{aligned}
$$

and we have used the definitions:

$$
\begin{aligned}
\phi_h^e &= \beta\ \xi_h^e + \alpha^2\ \psi_h^e && (42) \\
\chi_h^e &= \xi_h^e - \beta\ \psi_h^e && (43) \\
\upsilon_h^e &= \beta\ \xi_h^e + \alpha^2\ \psi_h^{-e} && (44) \\
\eta_h^e &= \xi_h^e - \beta\ \psi_h^{-e} && (45)
\end{aligned}
$$

At this point one could diagonalize the propagator but the diagonalization depends on k_0 and E_k and is quite complicated. It is not necessary to diagonalize the complete propagator for the purposes of this paper.

If we assume that only the γ_5 condensate contributes, this corresponds to the relations:

$$\xi^e_h \equiv \chi^e_h \equiv \beta\phi^e_h = e\ h\ \xi^+_+ \qquad \psi^e_h \equiv 0 \quad \upsilon^e_h \equiv \beta\xi^e_h \quad \eta^e_h \equiv \xi^e_h \tag{46}$$

and it can be shown that:

$$\begin{aligned} G^+ &= \sum_{e,h=-1,1} \frac{k_0 + e(E_k - e\ \mu)}{k_0^2 - (E_k - e\ \mu)^2 - (\xi^+_+)^2} P^e_h\ \gamma_0 \\ &= \sum_{e=-1,1} \frac{k_0 + e(E_k - e\ \mu)}{k_0^2 - (E_k - e\ \mu)^2 - (\xi^+_+)^2} \Lambda^e\ \gamma_0 \end{aligned} \tag{47}$$

which agrees with the quark propagator derived recently [22] in this ansatz. We will show below that while this ansatz is a good first approximation, the gap equation will not close under this ansatz and the fully general quark propagator is the more complex form shown above.

The analysis up until this point generalizes to perturbation theory analysis. For simplicity we restrict ourselves to NJL analyses in this paper.

For the purposes of this paper where the propagator will be used with an NJL interaction the roots simplify considerably. With foresight we can set

$$\psi^e_h = h\ \frac{\beta}{\alpha}\ \Delta_3 \tag{48}$$

$$\xi^e_h = h\ (e\Delta_1 - \alpha\Delta_3) \tag{49}$$

where Δ_1 and Δ_3 are defined in Appendix B. Δ_1 is the usual CFL condensate and Δ_3 is a new condensate. In this case we find that:

$$\det\Omega_+ = \det\Omega_- = \left(k_0^2 - (\epsilon^+)^2\right)\left(k_0^2 - (\epsilon^-)^2\right) \tag{50}$$

where:

$$(\epsilon^\pm)^2 = E_k^2 + \mu^2 + \Delta_1^2 + \Delta_3^2 \pm \sqrt{(2E_k\mu + 2\alpha\Delta_1\Delta_3)^2 + 4\beta^2\Delta_3^2\,(E_k^2 + \Delta_1^2)} \tag{51}$$

In the massless limit $\Delta_3 = 0$ [2], $\alpha = 0$, $E_k = |\vec{k}|$ and therefore the $\epsilon^\pm$ reduce to the those defined in [15].

[2] As we shall see later.

In the NJL model the full propagator is;

$$G^+(k) = \sum_{e,h=-1,1} \frac{\alpha\, e\, (k_0-\mu+e\, E_k))(k_0^2-(\omega_1)^e_+) - e(k_0-\mu-e\, E_k)\beta^3\Delta_1\Delta_3}{(k_0^2-(\epsilon^+)^2)\,(k_0^2-(\epsilon^-)^2)} P^e_h$$
$$-\frac{\beta/\alpha(e\, k_0 - e\,\mu - E_k)(k_0^2-(\omega_2)^e_+) + (k_0-\ \mu - e\, E_k)(\Delta_1 - e\Delta_3/\alpha)\Delta_3}{(k_0^2-(\epsilon^+)^2)\,(k_0^2-(\epsilon^-)^2)} Q^e_h$$

where:

$$(\omega_1)^e_h = (E_k + e\,\mu)^2 + (\Delta_1 + e\,\alpha\Delta_3)(\Delta_1 + e\,\Delta_3/\alpha) \quad (52)$$
$$(\omega_2)^e_h = (E_k + e\,\mu)^2 + \ (\Delta_1 + e\,\alpha\Delta_3)\Delta_1 \quad (53)$$

The quasiparticle propagator can now be used in the mean field gap equation[15]:

$$\Delta(k) = -ig^2 \int \frac{d^4q}{(2\pi)^4} \sum_{A,B} \bar{\Gamma}^A_\mu D^{\mu\nu}_{AB}(k-q) G^-_0(q)\Delta(q)G^+(q)\Gamma^B_\nu \quad (54)$$

$D^{\mu\nu}_{AB}(k-q)$ is the gluon propagator, Γ^B_ν is the interaction vertex, and:

$$\bar{\Gamma} = C\Gamma^T C^{-1}. \quad (55)$$

In the following sections we will present results for two different types of four fermion interactions. First we present results for the simplest NJL model motivated by the effective Lagrangian approach. Second we will present results for a four fermion interaction which has the color structure of a single gluon exchange.

The general quasiparticle propagator derived in this section could be used in a perturbation theory analysis of the gap equation. More importantly, the methods used above can be generalized to the physical case where the up and down quarks are essentially massless and the strange quark is massive.

4 Simplest NJL Model

The simplest possible four fermion interaction, motivated by the effective Lagrangian approach, is to take the interaction vertex to be $\Gamma^A_\mu = i$ and:

$$g^2 D^{\mu\nu}_{AB} \to \frac{1}{4} G\delta^{\mu\nu}\delta_{AB} \quad (56)$$

giving the matrix gap equation:

$$\Delta(k) = 8\; i\; G \int \frac{d^4 q}{(2\pi)^4} G_0^-(q) \Delta(q) G^+(q) \tag{57}$$

Acting on both sides of this equation with the operators γ_5, $\gamma_5\gamma_0$ and tracing over the spinor indices one can obtain the gap equations for Δ_1 and Δ_3 respectively. The color-flavor structure of the gap matrix can almost be ignored because the propagators do not have any color flavor structure and therefore the same factor will occur on both sides of the equation. However, if one is working in the degenerate three flavor case there is a slight complication to even the zero mass gap equation. We do not want to deal with this complication in this paper. One could think of this analysis as an approximate analysis in the three flavor case.

In the two flavor case there is only one color-flavor structure to be concerned about and this analysis is exact. This applies even to the case of the 2SC Color superconducting phase in the physical 3 flavor case as condensates involving the third color and flavor will simply decouple from the quantities that we are calculating.

The coupled gap equations are:

$$\begin{aligned} \Delta_1 &= 8\; i\; G \int \frac{d^4 q}{(2\pi)^4} \frac{2E_q \mu \alpha \Delta_3 + \Delta_1 (q_0^2 - E_q^2 - \Delta_1^2 + \Delta_3^2 - \mu^2)}{(q_0^2 - (\epsilon^+)^2)\,(q_0^2 - (\epsilon^-)^2)} \\ \Delta_3 &= -8\; i\; G \int \frac{d^4 q}{(2\pi)^4} \frac{2E_q \mu \alpha \Delta_1 + \Delta_3 (q_0^2 + \Delta_1^2 - \mu^2) + \Delta_3 (\beta^2 - \alpha^2)(E_q^2 - \Delta_3^2)}{(q_0^2 - (\epsilon^+)^2)\,(q_0^2 - (\epsilon^-)^2)} \end{aligned} \tag{58}$$

where terms linear in q_0 have been dropped since they will cancel out on integration over q_0 from $-\infty$ to ∞. The dependence of the condensate on momentum has been dropped since the right hand side of the gap equations are independent of $\vec{k}$.

Acting on (57) with the operators γ_0 and the identity and taking the trace leads to vanishing of the right hand side which is consistent with the assumption that Δ_4 and Δ_8 are zero. Acting with any of the other operators involving $\gamma \cdot \vec{k}$ will lead to a term involving $\hat{q} \cdot \hat{k}$ which will vanish by symmetry under the angular integration.

Evaluation of these equations can be facilitated by the analytic continuation $q_0 \to -iq_4$. The q_4 integration is then done by contour integration closing the contour in the upper half plane and picking up the poles at $i\epsilon^+$ and $i\epsilon^-$. The angular integrals can be done trivially giving:

$$\Delta_1 = \frac{G}{\pi^2}\int dq\, q^2 \left(\frac{\Delta_1}{\epsilon^+} + \frac{\Delta_1}{\epsilon^-} + \frac{(m\,\mu + \Delta_1\Delta_3)\Delta_3}{((\epsilon^+)^2-(\epsilon-)^2)\epsilon^+} - \frac{(m\,\mu + \Delta_1\Delta_3)\Delta_3}{((\epsilon^+)^2-(\epsilon-)^2)\epsilon^-}\right) \tag{59}$$

$$\begin{aligned}\Delta_3 \;&=\; \frac{G}{\pi^2}\int dq\, q^2 \left(\frac{\Delta_3}{\epsilon^+} + \frac{\Delta_3}{\epsilon^-} + \frac{m\,\mu\,\Delta_1}{((\epsilon^+)^2-(\epsilon-)^2)\epsilon^+} - \frac{m\,\mu\,\Delta_1}{((\epsilon^+)^2-(\epsilon-)^2)\epsilon^-}\right.\\ &+\; \left.\frac{(q^2+\Delta_1^2)\,\Delta_3}{((\epsilon^+)^2-(\epsilon-)^2)\epsilon^+} - \frac{(q^2+\Delta_1^2)\,\Delta_3}{((\epsilon^+)^2-(\epsilon-)^2)\epsilon^-}\right)\end{aligned} \tag{60}$$

If one assumes $\Delta_3 = 0$ the first equation reduces to the gap equation solved in [22]. The second equation does not vanish exactly under this assumption. If one takes $m = 0$ and $\Delta_3 = 0$ the second equation is trivially satisfied and the first equation becomes the gap equation for massless quarks.

The range of integration for q is not infinite since the NJL model is a four-fermion interaction model and must have an UV cutoff. The integrals are simply regularized by the factor:

$$\mathcal{R}(q) = \frac{\Lambda^4}{(q^2+\Lambda^2)^2}. \tag{61}$$

with $\Lambda = 1000$ MeV.

The gap equations were solved numerically for $\mu = 500$ MeV and $G = 48/\Lambda^2$ for different values of m. The results are shown in Figures (1) and (2). Also shown for comparison is the solution for Δ_1 where Δ_3 has been ignored.

One can see in Figure (2) that Δ_3 rises almost linearly with m and at $m = 150$ MeV is approximately 13% of Δ_1 at $m = 0$. In Figure (1) we can see that the inclusion of Δ_3 in the solution of the gap equations strongly affects the results for Δ_1. Δ_1 decreases with increasing m more rapidly if Δ_3 is not assumed to be zero. At $m = 150$ MeV Δ_1 is 14% less than $(\Delta_1)_{m=0}$ if Δ_3 is not neglected and only 5% less if it is.

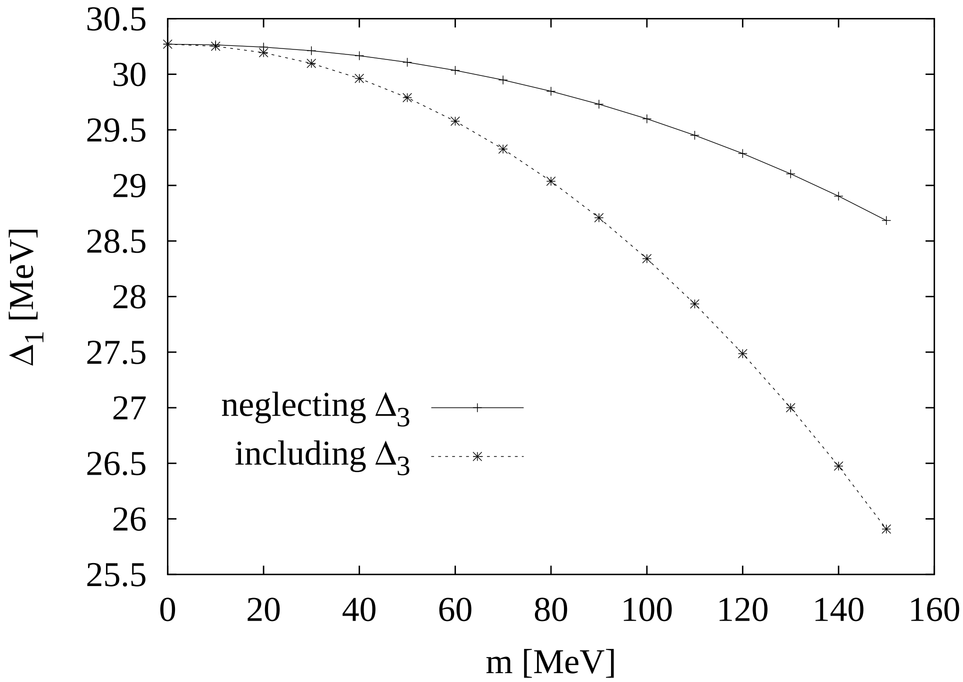

Figure 1: Numerical solutions for Δ_1 as a function of m for $\mu = 500$ MeV and $G = 48/\Lambda^2$ in the simplest NJL model. Shown are solutions neglecting the effect of Δ_3 and including the effect of Δ_3.

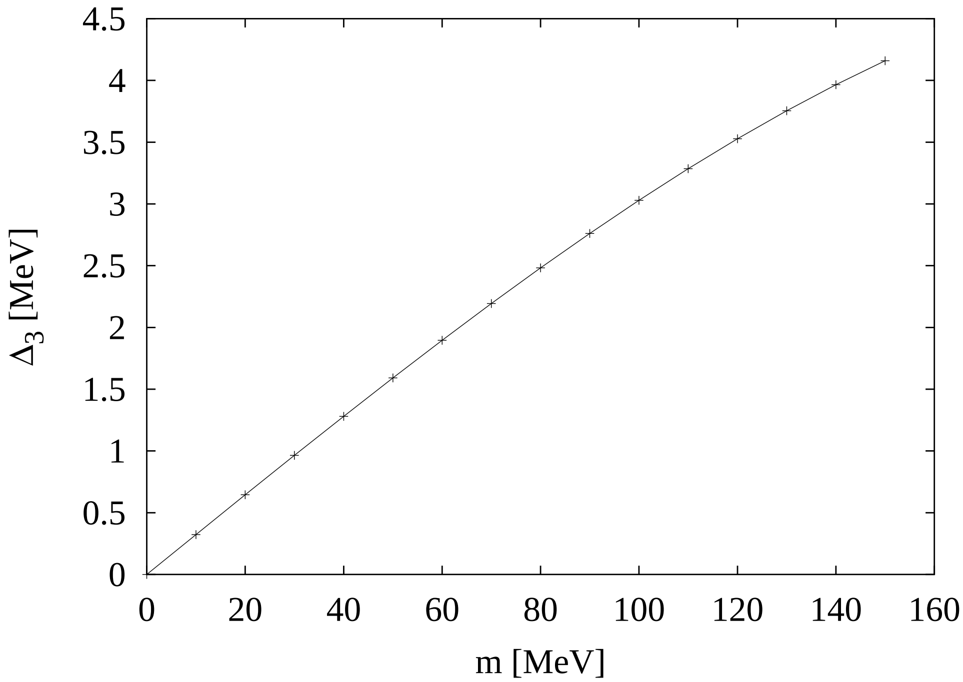

Figure 2: Numerical solutions for Δ_3 as a function of m for $\mu = 500$ MeV and $G = 48/\Lambda^2$ in the simplest NJL model.

The gap equations were solved numerically for $\mu = 500$ MeV and $G = 64/\Lambda^2$ for different values of m. The results are shown in Figures (3) and (4). Also shown for comparison is the solution for Δ_1 where Δ_3 has been ignored.

Again Figure (4) shows that Δ_3 rises linearly with m and at $m = 150$ MeV is approximately 12% of Δ_1 at $m = 0$. In Figure (3) we can see that again Δ_1 decreases with increasing m more rapidly if Δ_3 is not assumed to be zero. At $m = 150$ MeV Δ_1 is $\sim 7\%$ less than $(\Delta_1)_{m=0}$ if Δ_3 is not neglected and only $\sim 4\%$ less if it is.

One of the reasons for doing the analysis in the simplest NJL model is to estimate how the inclusion of Δ_3 might effect the results of [22]. With this goal in mind the gap equation has been solved in exactly the same approach as [22][3]. The results for $\mu = 400$ MeV are shown in Figures (5) and (6). The results in this case are very similar to the previous case. Δ_3 increases linearly reaching 18% of $(\Delta_1)_{m=0}$ at $m = 150$ MeV. Δ_1 is down at $m = 150$ MeV by $\sim 8.5\%$ and $\sim 5.5\%$ respectively including or neglecting Δ_3.

Extending this analysis out to quark mass of $m = 150$ MeV is partly for illustrative purposes but is also valuable for two other reasons.

First the analysis of [22] shows that in a coupled analysis of the superconducting (diquark) condensate and the axial condensate, constituent quark masses of the order of 100 MeV for the light quarks are possible at $\mu \approx 400$ MeV. For constituent quark masses of this order the results of our analysis show that the presence of a new condensate, Δ_3 which they neglected in their ansatz will be relevant. Their ansatz was a good first approximation as the new condensate and it's effects are not large. However, the effects are not neglible and the full analysis requires the more general ansatz and the use of the general quasiparticle propagator.

Second the methods used in this analysis can be extended to the physical case where the strange quark mass is of the order of 150 MeV. It is instructive therefore to carry out this analysis as a precursor to the analysis in the physical case.

[3] Their G_D does not correspond to our G.

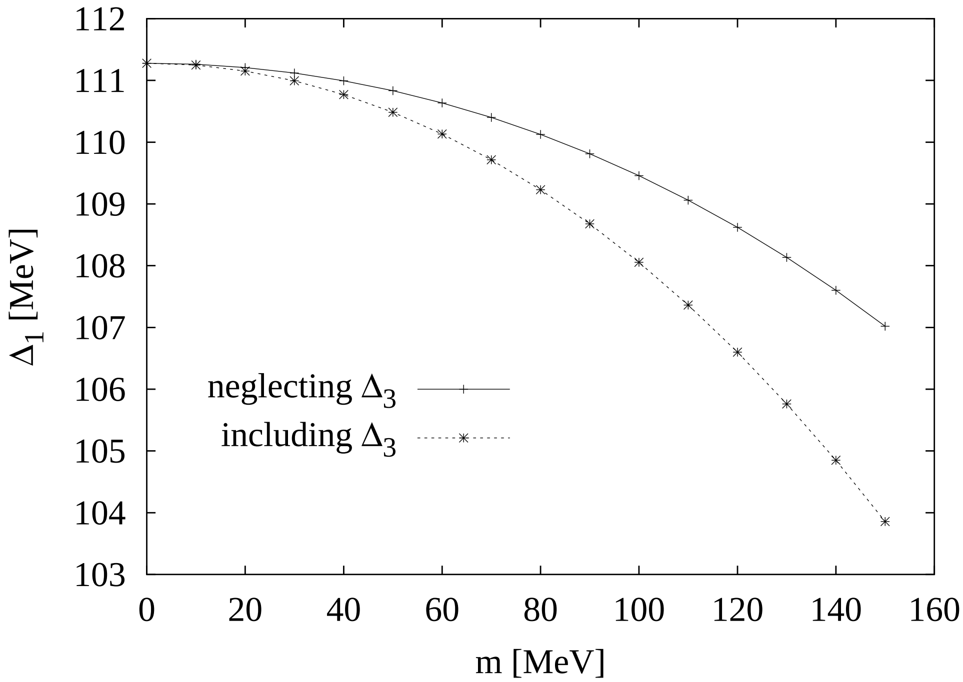

Figure 3: Numerical solutions for Δ_1 as a function of m for $\mu = 500$ MeV and $G = 64/\Lambda^2$ in the simplest NJL model. Shown are solutions neglecting the effect of Δ_3 and including the effect of Δ_3.

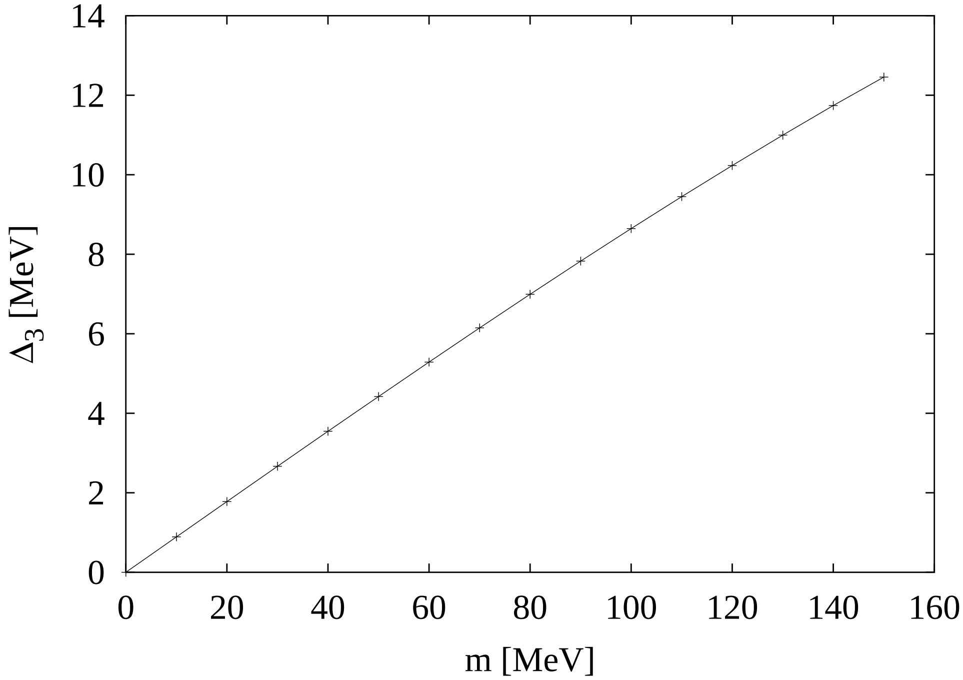

Figure 4: Numerical solutions for Δ_3 as a function of m for $\mu = 500$ MeV and $G = 64/\Lambda^2$ in the simplest NJL model.

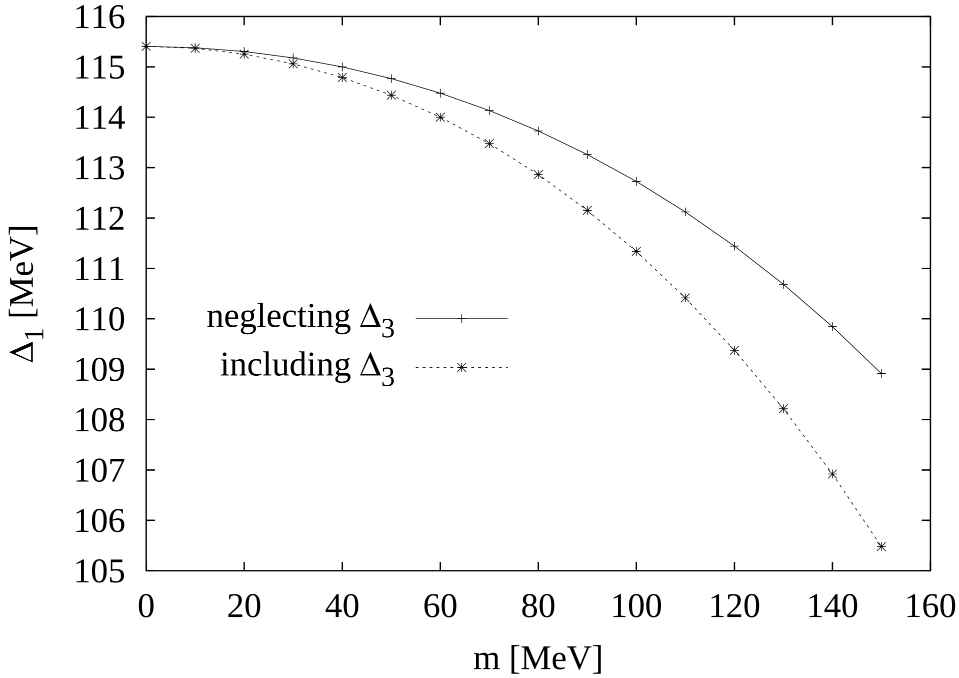

Figure 5: Numerical solutions for Δ_1 as a function of m for $\mu = 400$ MeV using the same approach as [22]. Shown are solutions neglecting the effect of Δ_3 and including the effect of Δ_3.

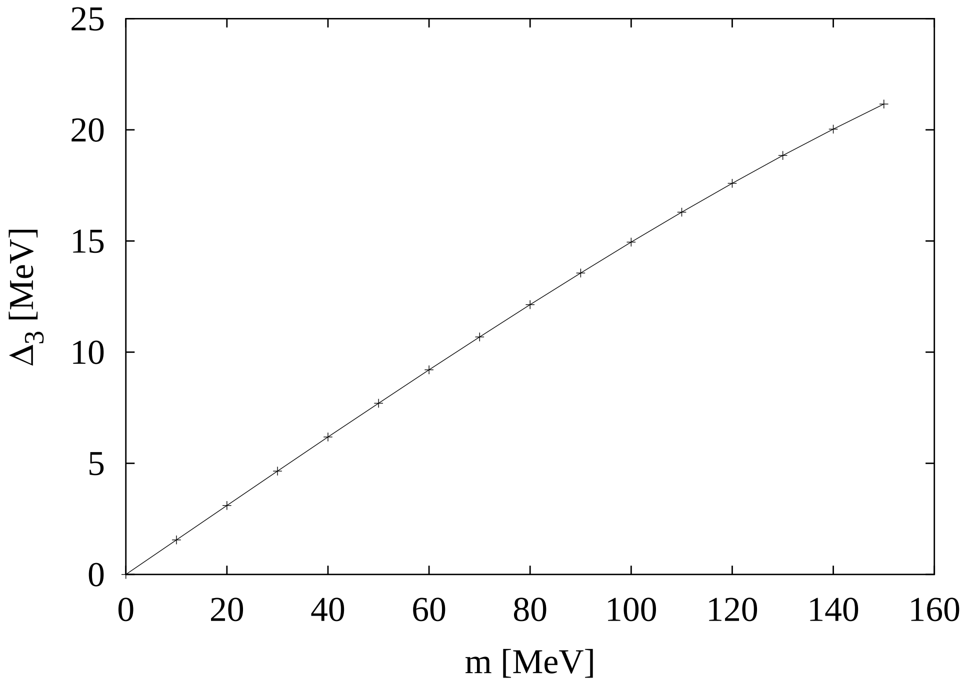

Figure 6: Numerical solutions for Δ_3 as a function of m for $\mu = 400$ MeV using the same approach as [22].

5 NJL Model motivated by Single Gluon Exchange

The fermion interaction which has the structure of single gluon exchange has the interaction vertex:

$$\Gamma^A_\mu = \gamma_\mu \frac{\lambda^A_c}{2} \qquad\qquad \bar{\Gamma}^A_\mu = -\gamma_\mu (\tfrac{\lambda^A_c}{2})^T \tag{62}$$

and:

$$g^2 D^{\mu\nu}_{AB} \to 3 G g^{\mu\nu} \delta_{AB} \tag{63}$$

giving the gap equation:

$$\Delta(k) = -2\; i\; G \int \frac{d^4 q}{(2\pi)^4}\; \gamma_\nu\; G^-_0(q) \Delta(q) G^+(q)\; \gamma^\nu \tag{64}$$

using the equations:

$$(\lambda^A)^T_{ij} \lambda^B_{jk} \lambda^A_{kl} = -\frac{8}{3} \lambda^B_{il} \quad \text{assuming } \lambda^B \text{ is antisymmetric} \tag{65}$$

If one multiplies on each side of this equation by γ_5 and $\gamma_5\gamma_0$, traces over the Dirac indices and uses the cyclicity of the trace and the relations:

$$\gamma_\mu \gamma^5 \gamma^\mu = -4\gamma^5 \tag{66}$$

$$\gamma_\mu \gamma^5 \gamma^0 \gamma^\mu = -\gamma^5 \gamma_\mu \gamma^0 \gamma^\mu = 2\gamma^5\gamma^0 \tag{67}$$

and then applies all the other machinery exactly as in the last section the coupled gap equations are obtained:

$$\Delta_1 = \frac{G}{\pi^2} \int dq\; q^2 \; \left(\frac{\Delta_1}{\epsilon^+} + \frac{\Delta_1}{\epsilon^-} + \frac{(m\;\mu + \Delta_1 \Delta_3)\Delta_3}{((\epsilon^+)^2 - (\epsilon-)^2)\epsilon^+} - \frac{(m\;\mu + \Delta_1\Delta_3)\Delta_3}{((\epsilon^+)^2 - (\epsilon-)^2)\epsilon^-} \right) \tag{68}$$

$$\begin{aligned} \Delta_3 \;\; &= \;\; -\frac{G}{2\pi^2} \int dq\; q^2 \; \left(\frac{\Delta_3}{\epsilon^+} + \frac{\Delta_3}{\epsilon^-} + \frac{m\;\mu\;\Delta_1}{((\epsilon^+)^2 - (\epsilon-)^2)\epsilon^+} - \frac{m\;\mu\;\Delta_1}{((\epsilon^+)^2 - (\epsilon-)^2)\epsilon^-} \right. \\ &+ \;\; \left. \frac{(q^2 + \Delta_1^2)\;\Delta_3}{((\epsilon^+)^2 - (\epsilon-)^2)\epsilon^+} - \frac{(q^2 + \Delta_1^2)\;\Delta_3}{((\epsilon^+)^2 - (\epsilon-)^2)\epsilon^-} \right) \end{aligned} \tag{69}$$

Notice that they are almost the same as the equations in the last section except for an overall factor of $-\frac{1}{2}$ in the second equation.

The gap equations were solved numerically for $\mu = 500$ MeV and $G = 48/\Lambda^2$ for different values of m. The results are shown in Figures (7) and (8). Also shown for comparison is the solution for Δ_1 where Δ_3 has been ignored.

One can see in Figure (8) that Δ_3 is negative and decreases linearly with m and at $m = 150$ MeV the magnitude is approximately 9% of Δ_1 at $m = 0$. In Figure (7) we can see that the inclusion of Δ_3 in the solution of the gap equations strongly affects the results for Δ_1. As shown in the last section, if Δ_3 is neglected Δ_1 decreases with increasing mass at $m = 150$ MeV and is down about 5% from the $m = 0$ result. By contrast if Δ_3 is included in the analysis, Δ_1 initially increases with m and then turns over at an m of around 90 MeV. The shift of Δ_1 is much smaller than when Δ_3 is neglected reaching an upward shift of $< 0.1\%$ and only down by 0.25% at $m = 150$ MeV.

The gap equations were solved numerically for $\mu = 500$ MeV and $G = 64/\Lambda^2$ for different values of m. The results are shown in Figures (9) and (10). Also shown for comparison is the solution for Δ_1 where Δ_3 has been ignored.

Again Figure (10) shows that Δ_3 is negative and decreases linearly with m and at $m = 150$ MeV the magnitude is approximately 6% of Δ_1 at $m = 0$. In Figure (9) we can see that in this case Δ_1 decreases with increasing m but decreases less rapidly if Δ_3 is not assumed to be zero. At $m = 150$ MeV, Δ_1 is $\sim 4\%$ less than $(\Delta_1)_{m=0}$ if Δ_3 is neglected and only $\sim 2\%$ less if it is not.

Neglecting Δ_3 in the solution of the gap equation gives a good approximation for Δ_1 in the NJL model with the structure of single gluon exchange. The inclusion of Δ_3 leads to non-negligible effects in the mass dependence of Δ_1. In the $N_f = 2$ case or, equivalently, the 2SC phase of 3 flavor color superconductivity, these effects will alter the location of the phase transition between the color superconducting phase and the normal phase. We can estimate that the effect on the phase transition will be of the same order or less than the effect on the Δ_1.

The complete analysis of the gap equations in this NJL model does not lead to drastic change in magnitudes of the gaps or the phase transitions but it does illustrate how to obtain the exact solutions in full generality. This is a significant advance.

This analysis is also useful as a precursor to the analysis in the physical case.

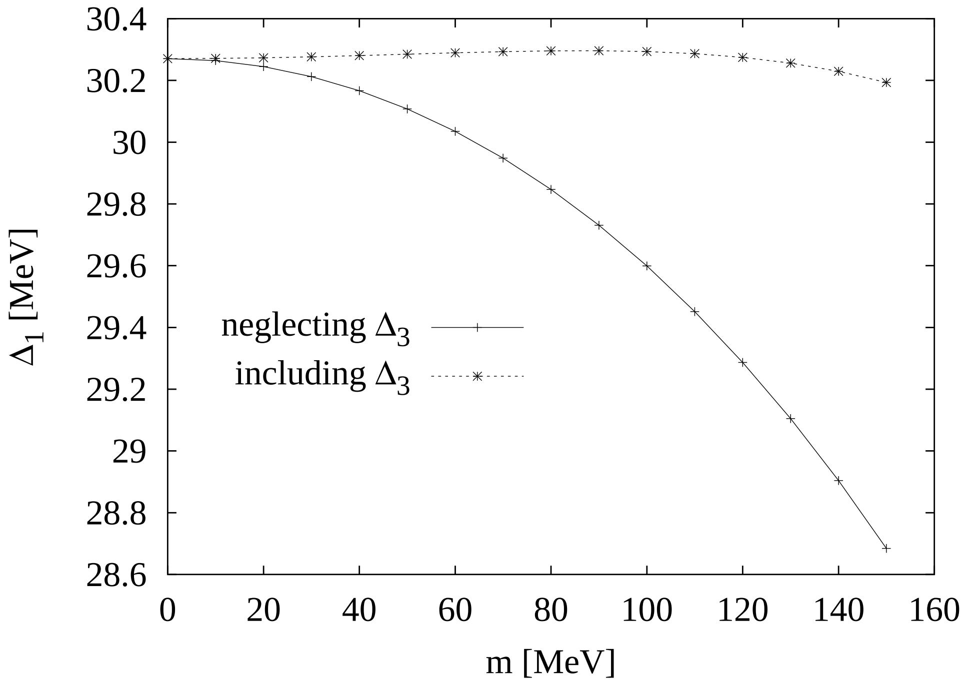

Figure 7: Numerical solutions for Δ_1 as a function of m for $\mu = 500$ MeV and $G = 48/\Lambda^2$ in the NJL model with the structure of single gluon exchange. Shown are solutions neglecting the effect of Δ_3 and including the effect of Δ_3.

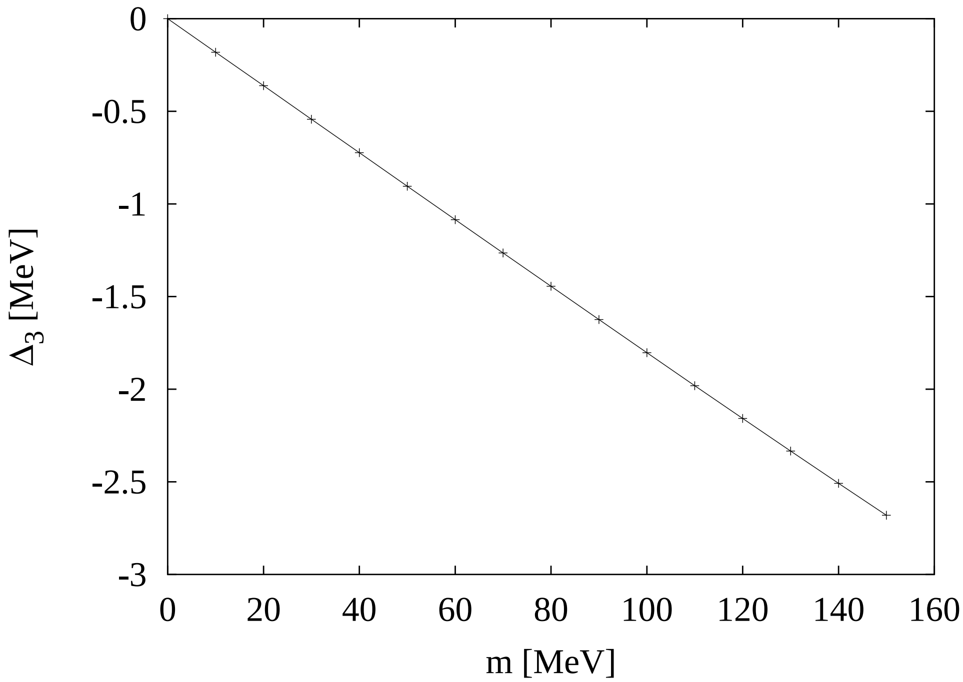

Figure 8: Numerical solutions for Δ_3 as a function of m for $\mu = 500$ MeV and $G = 48/\Lambda^2$ in the NJL model with the structure of single gluon exchange.

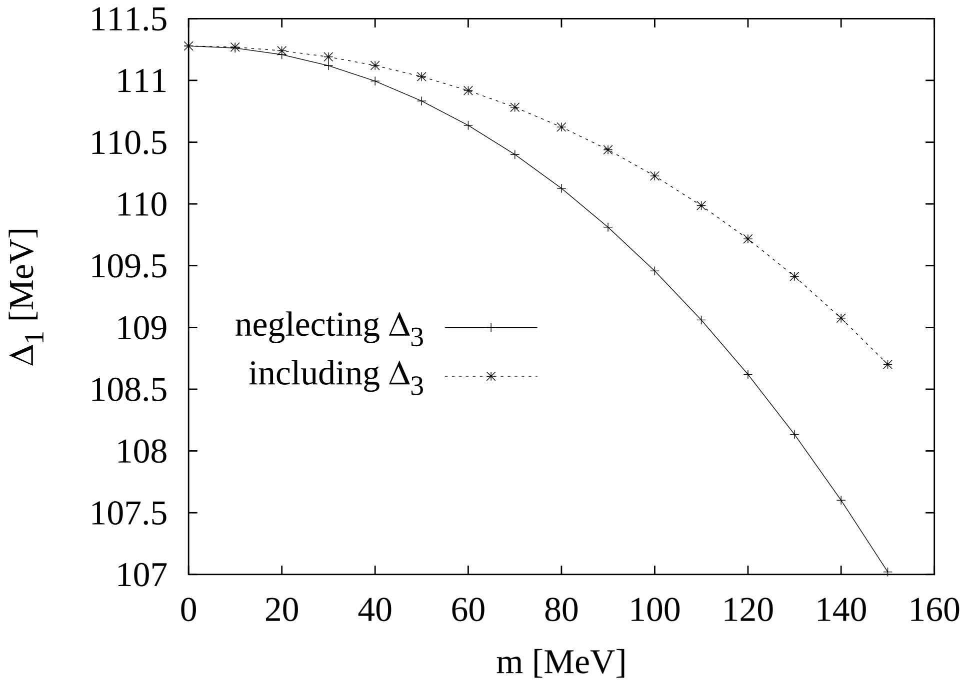

Figure 9: Numerical solutions for Δ_1 as a function of m for $\mu = 500$ MeV and $G = 64/\Lambda^2$ in the NJL model with the structure of single gluon exchange. Shown are solutions neglecting the effect of Δ_3 and including the effect of Δ_3.

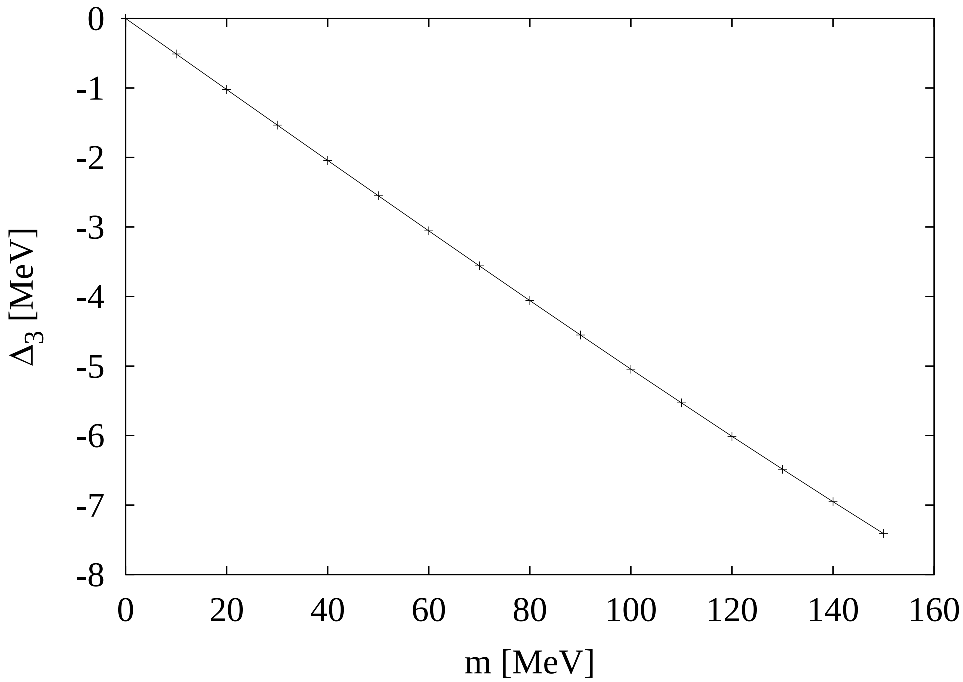

Figure 10: Numerical solutions for Δ_3 as a function of m for $\mu = 500$ MeV and $G = 64/\Lambda^2$ in the NJL model with the structure of single gluon exchange.

6 Conclusion

In this paper the completely general quasiparticle propagator for the case of equally massive quarks is derived. The quasiparticle propagator is then specialized to the propagator in an NJL model. It is shown that there is exactly one new condensate, Δ_3, in this model.

The quasiparticle propagator was used to numerically solve the gap equations for the CFL condensate, Δ_1, and the new condensate, Δ_3 in two NJL models as a function of the quark mass.

Results for Δ_1 and Δ_3 in the simplest NJL model were presented as a function of the quark mass. The results Δ_1 were compared with results where Δ_3 has been assumed to be zero. This was done for two different values of the four fermion coupling constant. The results show that the complete solution differs significantly from the approximate solution especially at large quark mass and smaller Δ_1.

Results in the same NJL model were also presented using the same approach and parameters as that in [22] to determine how much their results could be affected by neglecting Δ_3. The effect is not large but non-neglible and this analysis shows how to generalize their analysis to the complete case.

Results for Δ_1 and Δ_3 in the NJL model with the structure of single gluon exchange presented as a function of the quark mass. The results Δ_1 were compared with results where Δ_3 has been assumed to be zero. This was again done for two different values of the four fermion coupling constant. The results show that the complete solution differs significantly from the approximate solution especially at large quark mass and smaller Δ_1. Even for small values of m, if Δ_1 is small, the m-dependence of Δ_1 in the full analysis differs qualitatively from the approximate analysis.

Solving the color superconducting gap equations for the case of equal mass quarks is also valuable as a precursor to analyzing the physical case where the up and down quarks are essentially massless and the strange quark is massive. The methods used in this research combined with the methods of [8] can be combined to analyze the physical case where only the strange quark is given a non-zero mass which is the ultimate goal of this line of research.

7 Appendix A: Nambu-Gorkov Formalism

In the Nambu-Gorkov(cite?) approach one deals with the eight component spinors:

$$\Psi \equiv \left(\begin{array}{c} \psi \\ \psi_C \end{array} \right) \quad \bar{\Psi} \equiv (\psi \quad \psi_C) \tag{70}$$

with which the action can written concisely as:

$$I[\Psi, \bar{\Psi}] = \frac{1}{2} \int \bar{\Psi}\, \mathcal{S}^{-1}\, \Psi, \tag{71}$$

where:

$$\mathcal{S}^{-1} = \left(\begin{array}{cc} [G_0^+]^{-1} & \Delta^- \\ \Delta^+ & [G_0^-]^{-1} \end{array} \right) \tag{72}$$

and $\Delta^- \equiv \gamma_0 \Delta^+ \gamma_0$ and $G_0^\pm$ are the free quark (anti-quark) propagators.

The Grassman nature of the fermion fields and the off-diagonal terms in the action constrain[2, 15] the gap matrix in the massless case to be symmetric under the simultaneous interchange of all internal indices. This means that the gaps must be antisymmetric ($\bar{3}$) in both color and flavor or symmetric ($\bar{6}$) in both color and flavor and the resulting 9×9 gap matrix is symmetric.

The full propagator is:

$$\mathcal{S} = \left(\begin{array}{cc} G^+ & -G_0^+ \Delta^- G^- \\ -G_0^- \Delta^+ G^+ & G^- \end{array} \right). \tag{73}$$

8 Appendix B

The Bailin and Love motivated decomposition given in [15] is:

$$\Delta = \Delta_1 \gamma_5 + \Delta_2 \gamma \cdot \hat{k} \gamma_0 \gamma_5 + \Delta_3 \gamma_0 \gamma_5 + \Delta_4 + \Delta_5 \gamma \cdot \hat{k} \gamma_0 + \Delta_6 \gamma \cdot \hat{k} + \Delta_7 \gamma \cdot \hat{k} \gamma_5 + \Delta_8 \gamma_0 \tag{74}$$

where:

$$\Delta_1 \gamma_5 + \Delta_2 \gamma \cdot \hat{k} \gamma_0 \gamma_5 \tag{75}$$

represents condensation of fermions with the same chirality in the even parity channel,

$$\Delta_4 + \Delta_5 \gamma \cdot \hat{k} \gamma_0 \tag{76}$$

represents condensation of fermions with the same chirality in the odd parity channel.

$$\Delta_3\gamma_0\gamma_5 + \Delta_7\gamma\cdot\hat{k}\gamma_5 \tag{77}$$

represents condensation of fermions with opposite chirality in the even parity channel,

$$\Delta_6\gamma\cdot\hat{k} + \Delta_8\gamma_0 \tag{78}$$

represents condensation of fermions with opposite chirality in the odd parity channel.

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